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Two-Body Dirac Equation and Its WFO *

Hitoshi Ito

Department of Physics, Faculty of Science and Technology
Kinki University, Higashi-Osaka, 577 Japan

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Abstract

A relativistic equation is deduced for the bound state of two particles, by assuming a proper boundary condition for the propagation of the negative-energy states. It reduces to the (one-body)Dirac equation in the infinite limit of one of the constituent mass. It also has the symmetries to assure the existence of the anti-bound-state with the same mass. The interaction kernel(pseudo-potential) is systematically constructed by diagonalizing the Hamiltonian of the background field theory, by which the retardation effects are included in the interaction. Its wave function at the origin(WFO) behaves properly in a manner sugested by the covariant field theory.

1 Introduction

One of the unsatisfactory nature of the Bethe-Salpeter equation for the bound state is that it does not reduce to the Dirac equation in the infinite limit of the one of the constituent mass, when the interaction is assumed to be instantaneous[1]. We have to sum up all the crossed diagrams to recover the Dirac equation[2], which is impossible for the finite masses.

Historically, the relativistic single-time equation for the two-body system preceded the BS equation. Soon after the discovery of the Dirac equation, Breit proposed the equation of the form[3]

$$\{H_1(\boldsymbol{p}_1) + H_2(\boldsymbol{p}_2) + V\}\psi = E\psi, \tag{1}$$

where H_i is the Dirac Hamiltonian

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$$H_i(\boldsymbol{p}_i) = \boldsymbol{\alpha}_i \cdot \boldsymbol{p}_i + m_i\beta_i$$

and V is a local potential. The Breit equation reduces to the Dirac equation in the limit mentioned above but does not have the "E-parity symmetry", by which we mean that there is symmetric negative eigenvalue E for each positive one, which is interpreted as the bound state of the antiparticles.

E-parity symmetry is a consequence of the PCT invariance, for which we assume the non-quantized transformations. The Dirac Hamiltonian is odd under the PCT transformation and the interaction Hamiltonian considered below is invariant under it. We therefore see that the E-parity symmetry in the instantaneous BS equation is assured by a projection factor

$$\Lambda_{++} - \Lambda_{--} \tag{2}$$

in front of the potential V, which consists of the energy-projection operators

$$\Lambda_{\varepsilon\eta}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \Lambda^1_\varepsilon(\boldsymbol{p}_1)\Lambda^2_\eta(\boldsymbol{p}_2), \quad \varepsilon, \eta = + \text{ or } -, \tag{3}$$

where

$$\Lambda^i_\varepsilon(\boldsymbol{p}_i) = \{E_i(p_i) + \varepsilon H_i(\boldsymbol{p}_i)\}/2E_i(p_i),$$

$$E_i(p_i) = (\boldsymbol{p}_i^2 + m_i^2)^{1/2}.$$

It is necessary to introduce a similar factor in any attempt at the construction of an improved two-body equation.

The factor (2) comes from the Stückelberg-Feynman boundary condition for the propagation of the negative-energy state[4]. We will construct the equation for the unequal-mass constituents by imposing similar boundary condition and also investigate the equal-mass equation. But, before presenting it we should restrict the framework of consideration. For definiteness, we assume Abelian gauge fields interacting with the Dirac particles. We also work in the rest frame of the bound system($P = (E, \boldsymbol{0})$), since we are looking for the non-covariant approximation of the low-energy dynamics. $\boldsymbol{p}$ and $\boldsymbol{x}$, in the following, are the relative momentum and coordinate, respectively, in this frame.

2 Unequal-mass equation and its properties

By assuming a boundary condition for the propagation of the negative- energy states, we deduce a new equation, which we call the Two-body Dirac equation. We first consider the unequal-mass constituents and assume that the mass m_1 is larger than m_2.

2.1 *Two-body Dirac equation*

We start with the pseudo-4-dimensional form of the equation in the momentum space

$$\psi(p) = iS_F^{(2)}(P,p)\int V(\boldsymbol{p},\boldsymbol{q})\psi(q)d^4q/(2\pi)^4, \tag{4}$$

where $S_F^{(2)}(P,p)$ is the 2-body propagator in the lowest order and $V(\boldsymbol{p},\boldsymbol{q})$ represents the interaction, which is assumed not to depend on the relative energies but is not necessarilly an instanteneous local potential. We impose the boundary condition that the negative-energy states propagates backward in time. If we retain individuality of the constituents and use the usual Feynman propagator the factor (2) results. However, we can choose the other possibility in which we incorporate the idea that the bound two bodies should be treated as a quantum-mechanical unity.[1] Since m_1 is larger than m_2, the free part of the Hamiltonian has the same sign as that of the particle 1 in the CM system. Let us then modify the boundary condition as follows: *A bound two-particles state propagates backward in time if their net energy is negative.* By assuming it, we have the lowest propagator

$$S_F^{(2)}(P,p) = \sum_{\varepsilon\eta}\frac{1}{\lambda_1 E - p_0 - H_1(-\boldsymbol{p}) + i\varepsilon\delta}\frac{1}{\lambda_2 E + p_0 - H_2(\boldsymbol{p}) + i\varepsilon\delta}\Lambda_{\varepsilon\eta}\gamma_0^1\gamma_0^2, \tag{5}$$

where $\lambda_1 + \lambda_2 = 1$ and the limit $\delta \to +0$ is assumed.

After integrating out the redundant degree of the freedom in (4), we get the required equation

$$\{H_1(-\boldsymbol{p}) + H_2(\boldsymbol{p}) + \sum_{\varepsilon\eta}\varepsilon\Lambda_{\varepsilon\eta}V\}\psi = E\psi, \tag{6}$$

which reduces to the Dirac equation in the infinite limit of m_1. It is easy to see the E-parity symmetry of this equation.

If we would apply our equation to the scattering state, we shall have, from the time-dependent equation, the conserved probability density

$$\rho(\boldsymbol{x},t) = \psi(\boldsymbol{x},t)^\dagger\sum_{\varepsilon\eta}\varepsilon\Lambda_{\varepsilon\eta}\psi(\boldsymbol{x},t), \tag{7}$$

which is in accord with the boundary condition that the negative-energy state propagates backward in time carrying the negative probability density[5].[2] But, it does

[1] We can establish the concept of individuality in the quantum mechanics only through observation, which brings about a subtle point to the bound system even in infrared-free theories: To detect an individual one in the bound constituents, we need to separate them by applying the 3rd interaction, which inevitably destroys the original state. So there is no reason why we have to apply the free propagator individually to each constituent.

[2] We note that we can interprete the causal propagator of a Dirac particle, applied to one-particle wave function with the negative energy, also as carrying the negative probability density[6].

not necessarily provide the normalization condition for the bound state. For the scattering processes, the projected wave function $\Lambda_{\varepsilon\eta}\psi$ corresponds to the physical state of the (free)particles with the positive or negative energy E. And the above interpretation of the probability current actually says that *the state with the negative E is carrying the negative probability density*. However, for the bound state with the positive eigenvalue E, $\Lambda_{--}\psi$ or $\Lambda_{-+}\psi$ is merely *a negative-energy component in the representation in which the energy of the free particle is diagonal*. It is like a small component of the one-body Dirac equation.

Taking the above consideration into account, we restore the probability interpretation of the wave function and normalize it by assuming the probability density

$$\rho(\boldsymbol{x}) = \psi(\boldsymbol{x}, t)^\dagger \psi(\boldsymbol{x}, t). \tag{8}$$

Observables except for the Hamiltonian are self-adjoint under this metric:

$$(\phi, \hat{O}\psi) = (\hat{O}\phi, \psi). \tag{9}$$

The Hamiltonian is the operator ruling the time development of the system and is modified by the factor $\sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}$. Though it is not a self-adjoint operator, its eigenvalue is proved to be real if the inner product (13) below exists.

2.2 *Green's function and the vertex equation*

The Green's function G for (6) satisfies the operator equation

$$\{E - H_1 - H_2 - \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta} V\} G = \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}, \tag{10}$$

and

$$G\{E - H_1 - H_2 - V \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}\} = \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta}. \tag{11}$$

In the momentum representation, it can be written, by using the eigen-function $\chi_n(\boldsymbol{p})$ of (6), as

$$G(\boldsymbol{p}, \boldsymbol{p}') = \sum_n \frac{1}{N_n(E - M_n)} \chi_n(\boldsymbol{p}) \chi_n(\boldsymbol{p}')^\dagger + \text{continuum}, \tag{12}$$

where M_n is an eigenvalue and N_n is a normalization factor defined by

$$N_n = (\chi_n, \sum_{\varepsilon\eta} \varepsilon\Lambda_{\varepsilon\eta} \chi_n). \tag{13}$$

When one of the constituents(labeled with 2) is in the category of the antiparticle of the other, there can be an annihilation process for which the unamputated-decay-vertex Φ is given by

$$\Phi = C\gamma G, \tag{14}$$

where γ is the lowest vertex and C is the charge-conjugation matrix of the particle 2.

Φ satisfies the vertex equation

$$\Phi(E - H_1 - H_2 - V\sum_{\varepsilon\eta}\varepsilon\Lambda_{\varepsilon\eta}) = C\gamma\sum_{\varepsilon\eta}\varepsilon\Lambda_{\varepsilon\eta} \tag{15}$$

and the amputated vertex is

$$\Gamma = \gamma + C\Phi V. \tag{16}$$

We can determine the renormalization constant Z_1 for the wave function at the origin(WFO) from (15) and (16), if we need it[7].

2.3 *Interaction Hamiltonian*

We have, so far, not specified the interaction Hamiltonian(quasipotential). In this section, we investigate it for the one-(Abelian)guage-boson exchange in the Coulomb guage as an example. For the instantaneous part of the interaction, V is obvious. For the remaining part, we can specify the quasipotential in a clear way from the background field theory. We have already shown, for the Salpeter equation, that the quasipotential from the one-boson exchange is given through the diagonalization of Fukuda, Sawada, Taketani[8] and Okubo[9](FSTO)[10]. However, we first have to correct some error in Ref.[10].[3] Namely, we employed the usual Fock space and reinterpreted the matrix elements of the interaction Hamiltonian including the negative-energy indices as the one in this space. The guiding principle was the hole theory. But it brings the procedure into confusion, since we have revived the negative energy in Eq.(6). The correct choice is to generalize the Fock space by ignoring the hole theory. The hole theory is partially recovered afterward by including the projection factor in Eq.(6).

We show only the lowest one-boson exchange potential in the following. We first introduce the generalized Fock subspace of the free constituents, the bases of which are denoted by $|\varepsilon, \eta, \boldsymbol{p}\rangle$, where ε and η are the signs of the energies of the particles 1 and 2 respectively. We then diagonalize the Hamiltonian in the Schrödinger picture by using the FSTO method. The second-order boson-exchange potential in this subspace is given by

$$\begin{aligned}\langle\varepsilon, \eta, \boldsymbol{p}|V(1\mathrm{b})|\varepsilon', \eta', \boldsymbol{p}'\rangle = \frac{g^2}{(2\pi)^3}\sum_{ij}\alpha_{1i}(\delta_{ij} - \frac{1}{\boldsymbol{q}^2}q_iq_j)\alpha_{2j} \\ \times\frac{1}{2}[\frac{1}{\boldsymbol{q}^2 - \{\varepsilon E_1(p) - \varepsilon' E_1(p')\}^2} + \frac{1}{\boldsymbol{q}^2 - \{\eta E_2(p) - \eta' E_2(p')\}^2}],\end{aligned} \tag{17}$$

where $\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{p}'$. The retardation effects are included in this equation.

[3]The error is only conceptual for the Salpeter equation. The result of Ref.[10] is correct.

2.4 *WFO of the* $^1S_0$ *state*

Let us next study some fundamental feature of the equation. When it is applied to the system in which the pair annihilation of the constituents can occur, an important physical quantity is the wave function at the origin(WFO). For example, the decay amplitude of the pseudo-scalar $Q\bar{q}$ meson via a weak boson is proportional to the average WFO $\mathrm{Tr}\{\gamma_5\gamma_0\psi(0)\}$, where $\psi(0)$ is the charge conjugated(with respect to the particle $2(\bar{q})$) WFO. We investigate, in Appendix, the asymptotic behaviour of the momentum-space wave function by using the method given in Ref.[7]. We assume instantaneous exchange of a gauge boson[4]. The average WFO thus obtained is finite. This result is consistent with consideration on the covariant field theory. We note that the average WFO becomes divergent in the limit of the one-body Dirac equation, for which we have the renormalization procedure[11].

There are many "two-body Dirac equation" proposed. An interesting one from the point of view of the present paper is the one by Mandelzweig and Wallace[12]. Instead of redefining the two-body propagator, they intended to include the effects of the higher-order interaction (the crossed Feynman diagram) and obtained an equation which has the proper one-body limit and the E-parity symmetry. An important difference from our equation is in the average WFO considered above. It is finite in the Coulomb model but divergent if the transeverse part of the gauge-boson exchange is added[13].

2.5 *On the three-body equation*

The next comment to be made is on the three-body equation in which the interaction is assumed to be composed of two-body interactions. It is almost straight forward to deduce the three-body Hamiltonian. However, we have to examine how to define the projection factor in front of the quasipotential, since the sign of the energy of a interacting sub-(two body)system depends on the reference frame. This shows that the applicability of the single-time formalism is more restrictive in the three-body system than in the two-body system. The effective range of the velocity of the subsystem relative to the three-body CM system cannot be large. Under this restriction, we define the projection factor in the rest frame of the subsystem, which is, then, transformed into the rest frame of the three-body system. A part of the Hamiltonian which represent the interaction of the particle 1 and 2, for example, becomes

$$H_{12} = \sum_{\varepsilon} \varepsilon \Lambda^1_{\varepsilon}(\boldsymbol{p}_1) V_{12}(\boldsymbol{p}_1, \boldsymbol{p}_2), \tag{18}$$

when $m_1 > m_2$.

[4]The analysis in the Appendix cannot be applied to the retarded interaction.

Brown and Ravenhall pointed out that the three-body equation of some category[5] does not have proper solutions with the normalizable eigenfunction [15]. This phenomenon is called "continuum dissolution(CD)". The criteria for CD are as follows: (a) The equation decouples into the independent ones when a part of the interaction is switched off. (b) The subequations have solutions of the positive- and the negative-energy continua. Our three-body equation does not fulfill (a) and is free from CD.

3 Equal-mass equation

So far we have considered the unequal-mass equation. In this section we investigate the equal-mass limit of it, for which the exchange symmetry is an issue. We first note that the projection factor in front of the interaction term of (6) includes a part which violates this symmetry: It is shown that

$$\Lambda^{(V)} = \Lambda_{+-} - \Lambda_{-+} \tag{19}$$

and the Heisenberg's exchange operator

$$P_H = \frac{1}{4}(1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(1 + \boldsymbol{\rho}_1 \cdot \boldsymbol{\rho}_2)P_M \tag{20}$$

anticommute for $m_1 = m_2$, where the operator P_M exchanges the momenta (or coordinates). $\Lambda^{(V)}$ violates the symmetry since the remaining part of the Hamiltonian is commutable with P_H. For equal mass, we have two equations. One is the equation (6) with $m_1 = m_2$ and another is obtained by assigning a minus sign in front of $\Lambda^{(V)}$, which is the equal-mass limit of the equation with $m_2 > m_1$. It is the conjugate equation of (6) in the sence that (6) is converted into it by the transformation P_H. It is easy to show that these equations have the common eigenvalue spectrum: If an eigenfunction χ_n of (6) belongs to some eigenvalue M_n, $P_H\chi_n$ is the solution of the conjugate equation with the same eigenvalue. However, χ_n does not have the definite P_H-parity.

A way to recover the exchange symmetry is averaging the Hamiltonians of the equation (6) and the conjugate one. The resulting is the Salpeter equation. The eigenvalue of this equation is different from the above M_n. The difference is, however, small since it is of the 4th order in the symmetry-breaking part of the Hamiltonian. The quarkonium phenomenology for the equal-mass constituents is given in Ref.[10].

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[5]They considered the two-body equation in an external potential. The same argument is also applied to the three-body system [14].

Appendix

We examine the asymptotic ($p \to \infty$) behavior of the momentum-space wave function and show that the average WFO $\mathrm{Tr}\{\gamma_5\gamma_0\psi(0)\}/\sqrt{2}$ is finite.[6]

There are 4 partial amplitudes $h_{\varepsilon\eta}(p)$ in the $^1S_0$ state, with which the wave function is expanded as

$$\chi(\boldsymbol{p}) = \sum_{\varepsilon\eta}\sum_{r} c_r u^r_\varepsilon(-\boldsymbol{p}) v^{-r}_\eta(\boldsymbol{p}) h_{\varepsilon\eta}(p) (\frac{1}{16\pi E_1 E_2})^{1/2}, \tag{21}$$

where $c_{1/2} = -c_{-1/2} = 1/\sqrt{2}$ and the spinors u and v, for the particle 1 and 2 respectively, are defined in [7].

The average WFO for the annihilation decay through the axial-vector current is given by

$$\frac{1}{\sqrt{2}}\mathrm{Tr}\{\gamma_5\gamma_0\psi(0)\} = \frac{1}{\sqrt{8}\pi}\int (\frac{1}{E_1E_2})^{1/2} \times \sum_{\varepsilon\eta}\{\sqrt{(E_1+\varepsilon m_1)(E_2+\eta m_2)} - \varepsilon\eta\sqrt{(E_1-\varepsilon m_1)(E_2-\eta m_2)}\} h_{\varepsilon\eta}(p) p^2 dp. \tag{22}$$

We first assume the Coulomb potential. The partial-wave equation for the $^1S_0$ state is given by

$$\{E - \varepsilon E_1(p) - \eta E_2(p)\} h_{\varepsilon\eta}(p) = -\varepsilon\frac{\alpha}{4\pi}\sum_{\varepsilon'\eta'}\int dq \times \frac{q}{p}[\frac{1}{E_1(p)E_1(q)E_2(p)E_2(q)}]^{1/2}[\{A^1_{\varepsilon\varepsilon'}A^2_{\eta\eta'} + \varepsilon\varepsilon'\eta\eta' A^1_{-\varepsilon-\varepsilon'}A^2_{-\eta-\eta'}\}Q_0(z) + \{\varepsilon\varepsilon' A^1_{-\varepsilon-\varepsilon'}A^2_{\eta\eta'} + \eta\eta' A^1_{\varepsilon\varepsilon'}A^2_{-\eta-\eta'}\}Q_1(z)] h_{\varepsilon'\eta'}(q), \tag{23}$$

where $z = (p^2+q^2)/2pq$ and $Q_\ell(z)$ is the Legendre's function. $A^i_{\varepsilon\varepsilon'}$ is defined by

$$A^i_{\varepsilon\varepsilon'} = \sqrt{(E_i(p)+\varepsilon m_i)(E_i(q)+\varepsilon' m_i)}.$$

The asymptotic behavior of the wave function is determined from the integral region near the infinity. We then expand the both sides of (23) into the series of $1/p$ and $1/q$. We assume the power behavior of the wave function for large p. The independent amplitudes are chosen to be $h_A(p) \equiv \sum_\varepsilon h_{\varepsilon\varepsilon}(p)$, $h_B(p) \equiv \sum_\varepsilon \varepsilon h_{\varepsilon\varepsilon}(p)$, $h_C(p) \equiv \sum_\varepsilon h_{\varepsilon-\varepsilon}(p)$, and $h_D(p) \equiv \sum_\varepsilon \varepsilon h_{\varepsilon-\varepsilon}(p)$, which are expanded, in the high-momentum region, in power series of $1/p$:

$$h_X(p) = \sum_n C^n_X p^{-\beta_X - 2n - 1}.$$

[6] See Ref.[16] and references therein, for the Salpeter equation.

Integrals on the right-hand side can be done if we neglect infrared-divergent terms which are irrelevant to the leading asymptotic behavior. Now, we can determine the asymptotic indices β_X's from consistency[7]: We get, for h_A and h_B,

$$2C_A^0 p^{-\beta_A} - EC_B^0 p^{-\beta_B-1} = \frac{\alpha}{\pi} C_A^0 \frac{\pi}{1-\beta_A} \cot(\frac{\pi}{2}\beta_A) p^{-\beta_A} \tag{24}$$

and

$$2C_B^0 p^{-\beta_B} - EC_A^0 p^{-\beta_A-1} = \frac{\alpha}{\pi} C_B^0 \frac{\pi(1-\beta_B)}{\beta_B(2-\beta_B)} \tan(\frac{\pi}{2}\beta_B) p^{-\beta_B}, \tag{25}$$

where the terms of the higher power in $1/p$ are neglected. If we neglect the second term in the left-hand sides of (24), we find β_A in the range $1 < \beta_A < 2$ [7] and get $\beta_B = \beta_A + 1$ from (25). We obtaine another series by neglecting the second term in (25). For this, β_B is found to be in the range $2 < \beta_0 < \beta_B < 3$, where the lower bound β_0 corresponds to the upper bound $4/\pi$ of α above which the index β_A from (24) becomes complex. β_A of the second series is given by $\beta_A = \beta_B + 1$.

The asymptotic amplitudes h_C and h_D are determined dependently on h_A and h_B. We get, for the minimum indices

$$\beta_C = \min(\beta_A + 2,\ \beta_B + 1) \tag{26}$$

$$\beta_D = \beta_A + 1. \tag{27}$$

We see that the average WFO (22) is finite, because

$$\beta_B > 1 \ \text{ and } \ \beta_C > 2$$

hold for the asymptotic amplitudes. This conclusion is valid even if the instantaneous exchange(transverse part) of the gauge boson is added.

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